Review Chap 5: Credibility Theory

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目录

- Structure of the chapter
- Classical credibility theory
- Bayesian credibility theory
- Greatest accuracy credibility theory
- Empirical Bayes approach to credibility theory

Why Credibility Theory?

- Sometimes, experience data is very scarce
- Use the data in hand as well as the experience of others in determining rates and premiums
- How to balance current information and other collateral information?
- $$\mathbf{Z}\hat{\theta}_s + (1 - \mathbf{Z})\hat{\theta}_c$$

What's each section about?

- **Section 5.2**: According to frequentist approach, what is the necessary number of claims if we want to fully trust them? When inadequate samples are found, what should we do to balance current samples and other accessible data? **(Classical credibility theory)**
- **Section 5.3**: In Bayesian statistics, how can we update our knowledge of underlying parameters? **(Bayesian credibility theory)**
- **Section 5.4**: Posterior distribution may be difficult to determine, and the posterior mean may not be conveniently expressible as a linear combination of prior mean and sample mean. Is there a new theory? **(Greatest accuracy credibility theory / Bühlmann credibility)**
- **Section 5.5**: How can we estimate $E\left[s^2(\Theta)\right]$, $E[m(\Theta)]$ and $Var[m(\Theta)]$? **(Empirical Bayes approach to credibility theory)**

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Full credibility (Page 161 - 162)

- Limited fluctuation around real θ
- $$\text{Prob}\left[\left|\hat{\theta}_s - \theta\right| \leq k\theta\right] = \text{Prob}\left[\hat{\theta}_s - k\theta \leq \theta \leq \hat{\theta}_s + k\theta\right] \geq 1 - \alpha$$
- Compound Poisson, S:
$$r\lambda \geq \frac{z^2_{1-\alpha/2}}{k^2}\frac{E\left(X^2\right)}{E^2(X)} = \frac{z^2_{1-\alpha/2}}{k^2}\left[1 + cv^2(X)\right]$$
- Compound Poisson, N:
$$r\lambda \geq \frac{z^2_{1-\alpha/2}}{k^2}$$

Full credibility (Page 161 - 162)

- Compound binomial, S $(r = 1)$:

$$mq \geq \frac{z_{1-\alpha/2}^2}{k^2} \left[\frac{E\left(X^2\right)}{E^2(X)} - q \right]$$

- Compound binomial, N $(r = 1)$:

$$mq \geq \frac{z_{1-\alpha/2}^2}{k^2} (1 - q)$$

Partial credibility (Page 163 - 164)

Compound Poisson, S

$$P(|\mathbf{Z}\bar{S} - \mathbf{Z}\theta| \le k\theta) = 1 - \alpha$$

$$r\lambda = \mathbf{Z}^2 \frac{z_{1-\alpha/2}^2 E\left(X^2\right)}{k^2 E^2(X)} = \mathbf{Z}^2 n_{F(k,\alpha)} \text{ or } \mathbf{Z} = \sqrt{\frac{r\lambda}{n_{F(k,\alpha)}}}$$

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Bayesian Statistics (Page 164 - 165)

- Frequentist statistician: θ is fixed!
- Bayesian statistician: θ follows a distribution!
- Conditional density: $f_{\mathbf{X}|\Theta}(\mathbf{x} \mid \theta)$
- Prior(subjective): $f_{\Theta}(\theta)$
- When new information comes:

$$f_{\Theta|\mathbf{X}}(\theta \mid \mathbf{x}) = \frac{f_{\Theta,\mathbf{X}}(\theta, \mathbf{x})}{f_{\mathbf{X}}(\mathbf{x})} = \frac{f_{\mathbf{X}|\Theta}(\mathbf{x} \mid \theta) f_{\Theta}(\theta)}{\int f_{\mathbf{X}|\Theta}(\mathbf{x} \mid \theta) f_{\Theta}(\theta) d\theta}$$

- Given a particular loss function, we could get the Bayesian estimator

Normal | Normal model (Page 165 - 167)

- $X \mid \theta$ is $N\left(\theta, \sigma^2\right)$ where σ^2 is known
- $\theta \equiv \mu \sim N\left(\mu_0, \sigma_0^2\right)$
-
$$\begin{aligned} f_{\Theta|\mathbf{X}}(\theta \mid \mathbf{x}) &\propto e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{\left(\theta-\mu_0\right)^2}{2\sigma_0^2}} \\ &\propto e^{-\frac{\left(\theta-\theta^*\right)^2}{2\sigma^{*2}}}, \end{aligned}$$
- $\theta^* = \left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}\right) / \left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right)$ and $\sigma^{*2} = \left(\frac{1}{\sigma^2/n} + \frac{1}{\sigma_0^2}\right)^{-1}$

Normal | Normal model (Page 165 - 167)

- θ^* is a weighted average of prior mean μ_0 and sample mean $\bar{X}$
- $\theta^* = \dfrac{\sigma_0^2}{\sigma_0^2 + \sigma^2/n}\bar{X} + \dfrac{\sigma^2/n}{\sigma_0^2 + \sigma^2/n}\mu_0$
- $\mathbf{Z} = \dfrac{\sigma_0^2}{\sigma_0^2 + \sigma^2/n} = \dfrac{n/\sigma^2}{n/\sigma^2 + 1/\sigma_0^2} = \dfrac{n}{n + \sigma^2/\sigma_0^2}$
- $\mathbf{Z}$ is an increasing function of n for fixed σ_0^2 and σ^2
- The variance of the posterior can be made as small as desirable by taking a large enough sample size n

Poisson | Gamma model (Page 167 - 170)

- $\lambda \sim \Gamma(\alpha, \beta)$
- $$f_{\Lambda|\mathbf{x}}(\lambda \mid \mathbf{x}) \propto \frac{\prod_{j=1}^{n} \lambda^{x_j} e^{-\lambda}}{\prod_{j=1}^{n} x_j!} \frac{\beta^{\alpha} \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)}$$
  $$\propto \lambda^{\sum x_j + \alpha - 1} e^{-\lambda(n+\beta)}$$
- $\lambda \mid \mathbf{x} \sim \Gamma\left(\sum x_j + \alpha, n + \beta\right)$

Poisson | Gamma model (Page 167 - 170)

- Posterior mean (Bayesian estimator of λ with quadratic loss):

$$
\begin{aligned}
E(\Lambda \mid \mathbf{x}) &= \frac{\sum x_j + \alpha}{n + \beta} \\
&= \frac{n}{n+\beta} \frac{\sum x_j}{n} + \frac{\beta}{\beta + n} \frac{\alpha}{\beta} \\
&= \mathbf{Z}\bar{x} + (1 - \mathbf{Z})\mu_0
\end{aligned}
$$

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An old photo from about 10 years ago

图 1: Prof. Hans Bühlmann and Prof. Lianzeng Zhang

Basic Theory (Page 170-173)

- $$\min E\left(X_{n+1}-\left[a_0+\sum_{j=1}^{n} a_j X_j\right]\right)^2$$
- Let $a_j = a$

$$E\left(X_{n+1}-\left[a_0+a\sum_{j=1}^{n} X_j\right]\right)^2 = \left(na^2+1\right) E\left[s^2(\Theta)\right] + E\left[(na-1)m(\Theta)+a_0\right]^2$$

Bühlmann Credibility (Page 174-175)

- $$\begin{aligned} E\left(X_{n+1} \mid \mathbf{X} = \mathbf{x}\right) = & \frac{n}{n + E\left[s^2(\Theta)\right]/Var[m(\Theta)]}\overline{\mathbf{X}} \\ & + \left(1 - \frac{n}{n + E\left[s^2(\Theta)\right]/Var[m(\Theta)]}\right)E[m(\Theta)] \\ = & \mathbf{Z}\overline{\mathbf{X}} + (1 - \mathbf{Z})E[m(\Theta)], \end{aligned}$$
- $\mathbf{Z} = n/(n+K)$
- $K = E\left[s^2(\Theta)\right]/Var[m(\Theta)]$
- $E\left[s^2(\Theta)\right]$: Expected value of the **P**rocess **V**ariance (EPV)
- $Var[m(\Theta)]$: **V**ariance of the **H**ypothetical **M**eans (VHM)

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Motivation

- Calculation of $E[s^2(\Theta)]$, $E[m(\Theta)]$ and $Var[m(\Theta)]$ needs assumption on prior distribution (subjective)
- Can we estimate $E[s^2(\Theta)]$, $E[m(\Theta)]$ and $Var[m(\Theta)]$ from current samples?

Model 1 (Page 176-180)

- The data are of the form $\left\{ \left\{ X_{ij} \right\}_{i=1}^{N} \right\}_{j=1}^{n}$, where X_{ij} represents the aggregate claims in the j^{th} year from the i^{th} risk.
- $\bar{X}_i = \sum_{j=1}^{n} X_{ij}/n$ （各风险组内的样本均值）
- $\widehat{E[(\Theta)]} = \bar{X}$ （$\bar{X}_i$ 的平均）
- $E\left[\widehat{s^2(\Theta)}\right] = \sum_{i=1}^{N} \sum_{j=1}^{n} \left(X_{ij} - \bar{X}_i\right)^2 /(N(n-1))$ （各风险组内样本方差的平均）
- $\widehat{Var[m(\Theta)]} = \sum_{i=1}^{N} \left(\bar{X}_i - \bar{X}\right)^2 /(N-1) - \sum_{i=1}^{N} \sum_{j=1}^{n} \left(X_{ij} - \bar{X}_i\right)^2 /(nN(n-1))$ （各风险组内样本均值的样本方差，还需要经过 $E\left[\widehat{s^2(\Theta)}\right]/n$ 的调整）
- $$\frac{n}{n + E\left[s^2(\Theta)\right]/Var[m(\Theta)]}\overline{\mathbf{X}}_i + \left(1 - \frac{n}{n + E\left[s^2(\Theta)\right]/Var[m(\Theta)]}\right) E[m(\Theta)]$$

Model 2: Bühlmann-Straub Credibility (Page 180-183)

- Other relevant information like premium income or number of policies is given
- Using volume and samples to determine estimation of $E[s^2(\Theta)]$, $E[m(\Theta)]$ and $Var[m(\Theta)]$